A holographic counterpart of entanglement island

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Background:

JT gravity from dimension reduction

Entanglement entropy and defect extremal surface

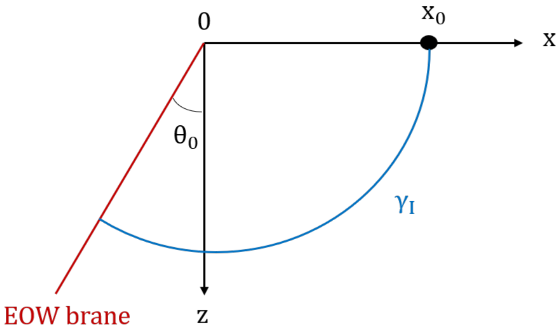
Overview:

- How to obtain 2d JT gravity from Karch-Randall brane-world setup.
- Defect extremal surface as the holographic counterpart of island formula.
  10.1007/JHEP02(2023)219 with Feiyu Deng and Yang Zhou

Background: AdS/BCFT correspondece

- For the holographic dual of boundary conformal field theory, it is given by the spacetime with brane.

$$I = \frac{1}{16\pi G_N}\int_N \sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_M \sqrt{-\gamma}K^{(\gamma)} + \frac{1}{8\pi G_N}\int_Q \sqrt{-h}K^{(h)} + I_Q + I_P \tag{1}$$



Background: AdS/BCFT correspondence

We impose Neumann boundary condition on brane Q:

$$K_{ab}^{(h)} - h_{ab}K^{(h)} = 8\pi G_N T_{ab} \tag{2}$$

Consider only tension term on brane:

$$I_Q = -\frac{1}{8\pi G_N}\int_Q \sqrt{-h}T, \tag{3}$$

$$K_{ab} - (K - T)h_{ab} = 0 \tag{4}$$

To match the symmetry, using metric ansatz:

$$ds^2 = d\rho^2 + l^2 cosh^2\frac{\rho}{l}\frac{-dt^2 + dy^2}{y^2} \tag{5}$$

Tension is related to the brane position

$$T = \frac{tanh(\rho_0/l)}{l} \tag{6}$$

Background: AdS/BCFT correspondece

- The coordinate transformation to Poincare coordinate is

$$z = y/\cosh\frac{\rho}{l}\ , \quad x = -y\tanh\frac{\rho}{l} \tag{7}$$

  we can also introduce θ, $\frac{1}{\cos(\theta)} = \cosh\left(\frac{\rho}{l}\right)$.

- Holographic entanglement entropy (RT formula):

$$S_I = \frac{\text{Area}\,(\gamma_I)}{4G_N} = \frac{c}{6}\log\frac{2x_0}{\epsilon} + \frac{c}{6}\,\text{arctanh}(\sin\theta_0) \tag{8}$$

  With second term as boundary entropy.

$$S_{bdy} = \log g \tag{9}$$

Background: AdS/BCFT correspondece

- Boundary g theorem from null energy condition on the brane
- The co-dimension two action I_P does not matter in most cases, but does influence one physical quantity, stress energy tensor. Throughout our talk, it doesn't matter.
- AdS/BCFT setup only considered classical matter (tension) on it. If we instead add quantum field on it, it becomes a defect brane.

  For the boundary, the BCFT becomes d(defect) CFT.

$$K_{ab} - (K - T)h_{ab} = 8\pi G_N \langle T_{ab} \rangle \tag{10}$$

JT gravity:

- JT gravity is 2d dilaton gravity with linear potential.

$$
\begin{aligned}
I = &-\frac{\phi_0}{16\pi G}[\int \sqrt{g}R + 2\int_{bdy} K] \\
&- \frac{1}{16\pi G}[\int d^2x\sqrt{g}\phi(R+2) + 2\int_{bdy} \phi_b K]
\end{aligned}
\tag{11}
$$

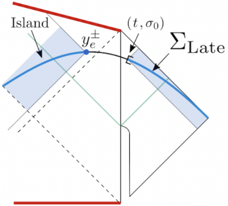
- The on shell action is governed by the boundary trajectory which is Schwarzian action.
- It is the holographic dual to the SYK model in the low energy limit.
- The black hole information paradox is first elucidated in JT gravity coupled to bath setup.
- Actually, the full gravitational path integral can be calculated in JT gravity, so it is the first quantum gravity. arXiv:1903.11115

Island formula:

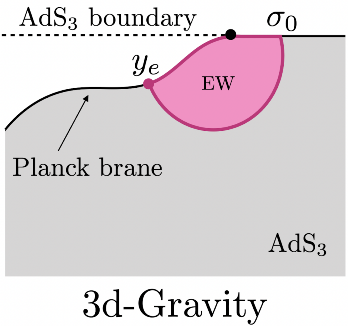
- In 2019, there is a leap forward towards understanding black hole information paradox. We can consistently calculate the fine-grained entropy of Hawking radiation which follows the page curve. It is now called entanglement island.

$$S = \min_{X} \left\{ \mathrm{ext}_X \left[\frac{\mathrm{Area}(X)}{4G_N} + S_{\text{semi-cl}} \ (\Sigma_X) \right] \right\}, \qquad (12)$$

- Based on entanglement wedge reconstruction proposal, black hole interior is encoded in the radiation.



- Island formula comes from the new saddle point called "Replica wormhole" in the Euclidean path integral when computing EE.
- Maldacena etal propose the so-called double holography picture, and interpret the island formula from higher dimensional point of view.



- In this paper, by combining braneworld and AdS/CFT duality, we provide an alternative understanding which is different from original double holography prescription.

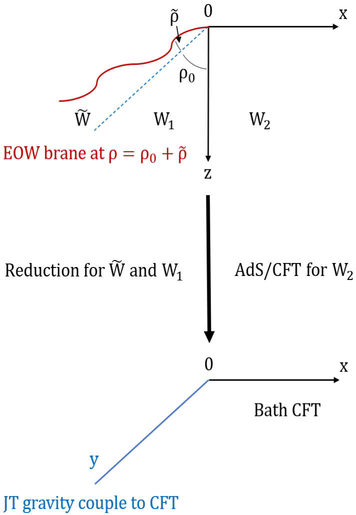
Partial dimension reduction:

- The 2d picture is the JT gravity+CFT model coupling to bath CFT.
- **Double holography picture:** 2d CFT is dual to some higher dimensional bulk, CFT and gravity comes from some kind of RS reduction.

$$\int D[g^{(3)}_{ab}]e^{-S_{grav}} = \int D[g^{(2)}_{\mu\nu}]D[\Phi_{CFT}]e^{-S(\Phi_{CFT},g^{(2)}_{\mu\nu})} \qquad (13)$$

- **Partial reduction:** we choose zero tension brane as the dividing line. On the right we use the Z_2 quotient of AdS/CFT and on the left we invoke the dimension reduction.
- 2d dilaton gravity comes from 3d gravity, while CFT is not from gravity but from the properties that we treat the brane as a defect with quantum field localized on it.
- Below, we will use JT as an example to support our partial reduction setup.

Derive JT gravity by dimension reduction:



$$ds^2 = d\rho^2 + l^2 cosh^2 \frac{\rho}{l} \tilde{h}_{ab} dx^a dx^b \tag{14}$$

The brane Q is denoted as: $\rho = \rho_0 + \tilde{\rho}(x^a)$
Bulk Ricci scalar has a simple relation:

$$\sqrt{-g}R = \sqrt{-g^{(2)}}[R^{(2)} - \frac{2(3cosh^2\frac{\rho}{l} - 1)}{l^2 cosh^2\frac{\rho_0}{l}}] \tag{15}$$

where

$$g^{(2)}_{ab} = l^2 cosh^2\frac{\rho_0}{l}\tilde{h}_{ab} \tag{16}$$

The bulk action after reduction is

$$\frac{1}{16\pi G}\int_{W_1+\tilde{W}}\sqrt{-g}(R-2\Lambda) = \int\frac{\rho_0+\tilde{\rho}}{16\pi G_N}\sqrt{-g^{(2)}}R^{(2)} - \frac{1}{16\pi G_N}\int\frac{\sinh(\frac{2(\rho_0+\tilde{\rho})}{l})}{l\cosh^2\frac{\rho_0}{l}}\sqrt{-g^{(2)}} \tag{17}$$

Extrinsic curvature of the fluctuating brane:

$$K = \nabla_\alpha n^\alpha = \frac{2\tanh(\frac{\rho_0}{l} + \frac{\tilde{\rho}}{l})}{l} - \frac{1}{l^2\cosh^2\frac{\rho}{l}\sqrt{-\tilde{h}}}\partial_a(\sqrt{-\tilde{h}}\tilde{h}^{ab}\partial_b\tilde{\rho}), \tag{18}$$

The Gibbons-Hawking boundary term reads

$$\begin{aligned}\frac{1}{8\pi G_N}\int_Q \sqrt{-h}(K - T) = {} & \frac{1}{8\pi G_N}\int \frac{\sinh(\frac{2\rho_0+2\tilde{\rho}}{l})}{l\cosh^2\frac{\rho_0}{l}}\sqrt{-g^{(2)}} \\ & - \frac{1}{8\pi G_N}\int \frac{\tanh\frac{\rho_0}{l}\cosh^2(\frac{\rho_0+\tilde{\rho}}{l})}{l\cosh^2\frac{\rho_0}{l}}\sqrt{-g^{(2)}}\end{aligned} \tag{19}$$

Where $h_{ab} = l^2\cosh^2\frac{\rho}{l}\tilde{h}_{ab}$, Combine bulk and boundary term together , we get (expand over $\tilde{\rho}/\rho_0$)

$$I_{tot} = \frac{\rho_0}{16\pi G_N}\int \sqrt{-g^{(2)}}R^{(2)} + \frac{\rho_0}{16\pi G_N}\int \sqrt{-g^{(2)}}\frac{\tilde{\rho}}{\rho_0}(R^{(2)} + \frac{2}{l^2\cosh^2\frac{\rho_0}{l}}) \tag{20}$$

After identifying 2d effective Newton constant $G_N/\rho_0 = G_N^{(2)}$, and $\frac{\tilde{\rho}}{\rho_0} = \phi$. It is the bulk JT gravity action.
Starting from 2d action we get from dimension reduction, varying $\tilde{\rho}$

$$R^{(2)} = -\frac{2}{l^2 \cosh^2 \frac{\rho_0}{l}} \tag{21}$$

Varying $g^{(2)}$, we get

$$(\nabla_\mu \nabla^\mu - 2)\phi = 0 \tag{22}$$

We can take the static solution of the dilaton equation of motion:

$$\phi = \frac{\tilde{\rho}}{\rho_0} = \frac{\bar{\phi}_r}{y} \tag{23}$$

Alternative interpretation of the dilaton from 3d bulk:

What is the dilaton in JT? Fluctuation of the brane !
Choosing the gauge:

$$\bar{\rho} = \rho + \phi(x^\mu) \tag{24}$$

ρ with fluctuation and $\bar{\rho}$ without fluctuation

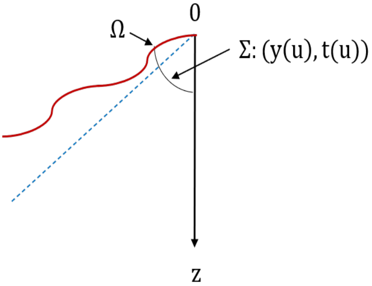
$$ds^2 = d\bar{\rho}^2 + (\cosh^2 \frac{\bar{\rho}}{l} \gamma_{\mu\nu} + \bar{h}_{\mu\nu}) d\bar{x}^\mu d\bar{x}^\nu \tag{25}$$

$\bar{h}_{\mu\nu}$ is small , expand the Neumann boundary condition

$$K_{\mu\nu} - (K - T) h_{\mu\nu} = 0 \tag{26}$$

in terms of $\bar{h}_{\mu\nu}$, first order equation for $\bar{h}_{\mu\nu}$ gives the dilaton equation of motion.
Dilaton equation of motion is from Neumann boundary condition.
Dilaton in JT is the brane bending mode appeared in arXiv:2205.15500

Next we obtain boundary Schwarzian action from the 3d asymptotic boundary action.



Asymptotic boundary $(y(u), t(u))$.

$$ds^2 = d\rho^2 + l^2 \cosh^2(\frac{\rho}{l})(\frac{y'^2 du^2 - t'^2 du^2}{y^2}), \qquad (27)$$

For UV cutoff surface:

$$T^{\nu} = (t', y', 0) \tag{28}$$

$$n_{\mu} = \left(\frac{y' l \cosh\frac{\rho}{l}}{y\sqrt{t'^2 - y'^2}}, -\frac{t' l \cosh\frac{\rho}{l}}{y\sqrt{t'^2 - y'^2}}, 0\right) \tag{29}$$

The extrinsic curvature at the asymptotic boundary is

$$K_{\Sigma} = g^{\mu\nu}\nabla_{\mu} n_{\nu} = \frac{1}{l\cosh\frac{\rho}{l}} \frac{t'^3 + yy't'' - t'y'^2 - t'yy''}{(t'^2 - y'^2)^{3/2}} \tag{30}$$

ρ direction can be integrated.

$$I_{bJT} = \frac{1}{8\pi G_N}\int_{\Sigma}\sqrt{-\gamma}K_{\Sigma} = \frac{\rho_0}{8\pi G_N}\int du\sqrt{\frac{t'^2 - y'^2}{y^2}}\left(1 + \frac{\tilde{\rho}}{\rho_0}\right)K_{\Omega} \tag{31}$$

Fixing boundary metric to be

$$g|_{bdy} = -\frac{l^2\cosh^2\frac{\rho}{l}}{\epsilon^2} \tag{32}$$

Schwarzian:

Solving

$$\frac{t'^2 - y'^2}{y^2} = \frac{1}{\epsilon^2} \tag{33}$$

to leading order $y = \epsilon t' + O(\epsilon^2)$

$$K_\Omega = 1 - \epsilon^2 \mathrm{Sch(t(u), u)} \tag{34}$$

$$\mathrm{Sch}(t(u), u) = \frac{2t' t''' - 3t''^2}{2t'^2}. \tag{35}$$

Varying with respect to the metric in the bulk action, we can solve the dilaton profile

$$\frac{\tilde{\rho}}{\rho_0} = \frac{\bar{\phi}_r}{y}. \tag{36}$$

$$I_{bJT} = \frac{\rho_0}{8\pi G_N} \int \frac{du}{\epsilon} K_\Omega + \frac{\rho_0}{8\pi G_N} \int \frac{du}{\epsilon^2} \phi_r(u) K_\Omega. \tag{37}$$

- First term is the topological term, second term is

$$\frac{\rho_0}{8\pi G_N}\int \frac{du}{\epsilon^2}\phi_r(u)(1-\epsilon^2\mathrm{Sch}(t(u),u)), \tag{38}$$

- Note that when calculating K_Q, there is total derivative term, the total derivative term gives $-\frac{\rho_0}{8\pi G_N}\int du\frac{\phi_r(u)}{\epsilon^2}$ by stokes theorem.
  So the only remaining action of JT gravity is Schwarzian action.

Next I will calculate entanglement entropy, before that let us review how to compute entanglement entropy of an interval in 2d CFT.

In 2d CFT the EE of $[x_1, x_2]$ is equivalent to correlation function of twist operator.

$$S = \lim_{n \to 1} \frac{1}{1-n} log \langle \sigma(x_1, \bar{x}_1) \sigma(x_2, \bar{x}_2) \rangle \tag{39}$$

The correlation function transforms as

$$\langle \sigma(x_1, \bar{x}_1) \sigma(x_2, \bar{x}_2) \rangle_{\Omega^{-2} g} = \Omega(x_1, \bar{x}_1)^{d_n} \Omega(x_2, \bar{x}_2)^{d_n} \langle \sigma(x_1, \bar{x}_1) \sigma(x_2, \bar{x}_2) \rangle_g \tag{40}$$

Where conformal dimension $d_n = \frac{c}{12}(n - \frac{1}{n})$.

So in conformal flat spactime

$$S \to S_{flat} - \frac{c}{6} \log \Omega(x) \tag{41}$$

For conformal flat spacetime

$$S_{CFT}(x_1, x_2) = \frac{c}{6} log(\frac{|x_1 - x_2|^2}{\epsilon_1 \epsilon_2 \Omega(x_1, \bar{x_1})\Omega(x_2, \bar{x_2})}) \tag{42}$$

Ω is the conformal factor defined by $ds^2 = \Omega^{-2} ds^2_{flat}$.
For BCFT in conformal flat spacetime, EE in $[0, y_1]$:

$$S = lim_{n\to 1} \frac{1}{1-n} log\langle \Phi_n(y_1)\rangle_Q \tag{43}$$

$$\langle \Phi_n(y)\rangle_Q = \Omega^{d_n} \langle \Phi_n(y)\rangle_{flat} \tag{44}$$

For interval $[0, y]$, the one point function is

$$\langle \Phi_n(y)\rangle_{flat} = \frac{g_n}{|2y/\epsilon_y|^{d_n}} \tag{45}$$

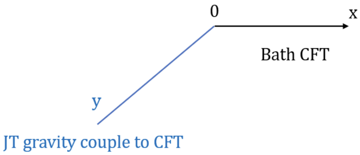
Entanglement entropy: QES(island)

In this slide, we review the QES formula in JT:

$$S = min\{ext_X[\frac{Area(X)}{4G_N} + S_{semi}(\Sigma_X)]\} \tag{46}$$

For 2d effective field theory:

$$\begin{aligned} I_{\mathrm{2d}} &= \frac{1}{16\pi G_N^{(2)}}\int\sqrt{-g^{(2)}}R^{(2)} \\ &+ \frac{1}{16\pi G_N^{(2)}}\int\sqrt{-g^{(2)}}\frac{\tilde{\rho}}{\rho_0}\left(R^{(2)} + \frac{2}{l^2\cosh^2\frac{\rho_0}{l}}\right) + I_{\mathrm{CFT}}. \end{aligned} \tag{47}$$



The island formula gives the entropy for $[0, L]$:

$$S_{gen}(a) = \frac{1}{4G_N^{(2)}}(1 + \frac{\tilde{\rho}}{\rho_0}) + \frac{c}{6} \log \frac{(L+a)^2 l}{a \cos\theta_0 \epsilon \epsilon_y} \tag{48}$$

$\frac{a\cos\theta_0}{l}$ is the warp factor for point a.
By solving the $\partial_a S_{gen}(a) = 0$, we get the position of the QES.

$$a = \frac{L}{2}[\sqrt{36\mu^2 + 36\mu + 1} + 6\mu + 1] \tag{49}$$

Where $\mu = \frac{\rho_0 \bar{\phi}_r}{6lL} = \frac{1}{6}\frac{\tilde{\rho}(a)}{l}\frac{a}{L}$. There is an interesting relation:

$$\frac{a}{L} = \frac{1 + \tilde{\rho}/l}{1 - \tilde{\rho}/l} \tag{50}$$

This is the calculation in Maldacena's paper: "island outside the horizon, arXiv: 1910.11077. "

Entanglement entropy: DES

If we modify BCFT to be defect CFT, there is matter field on the brane. As we consider vacuum state, the stress energy tensor 1pt function is from trace anomaly term.

$$\langle T_{ab} \rangle_{AdS} = \chi h_{ab} \tag{51}$$

This behaves like tension so we just include this into tension term in e.o.m.

But it contributes non-trivial entropy.

Quantum corrected RT surface:

$$S_{DES} = min_{\Gamma} \{ ext_{\Gamma, X} [\frac{Area(\Gamma)}{4G_N} + S_{defect}(D)]\} \tag{52}$$

RT surface is still a part of geodesic thanks to the matter field localized on the brane.

Using embedding formalism, we find that the geodesic distance betweeen $A = (t, a\cos\theta, -a\sin\theta)$ and $B = (t, \epsilon, L)$ is

$$\frac{Area(\Gamma)}{4G_N} = \frac{l}{4G_N}\mathrm{arccosh}\left(\frac{(L + a\sin\theta_0)^2 + a^2\cos^2\theta_0}{2a\cos\theta_0\epsilon}\right) + \frac{\rho_0\bar{\phi}_r}{4G_N a} \tag{53}$$

The defect entropy contribution is

$$S_{defect}(D) = \frac{c'}{6} log \frac{2l}{\epsilon_y \cos\theta_0} \tag{54}$$

Entanglement entropy: DES

$$\partial_a S_{gen}(a) = 0 \tag{55}$$

This leads to the position

$$a = \frac{3^{2/3} L(12\mu^2 + 12\mu \sin\theta_0 + 1)}{3\nu} + 2\mu L + \frac{L\nu}{3^{2/3}} \tag{56}$$

Where

$$\nu = (72\mu^3 + \frac{1}{6}\sqrt{\gamma} + 108\mu^2 \sin\theta_0 + 36\mu)^{1/3} \tag{57}$$

$$\gamma = 46656\mu^2(2\mu^2+3\mu\sin\theta_0+1)^2-108(12\mu^2+12\mu\sin\theta_0+1)^3 \tag{58}$$

After that we can compare between EE computed using QES and defect extremal surface.

Equivalence between DES and QES:

- $\rho_0/l \gg 1$, μ can be $O(1)$, expand in $\omega = \frac{\pi}{2} - \theta_0$.

$$a_{DES} = a_{island}\,, \quad S_{DES} = S_{island} \tag{59}$$

- ρ_0/l finite, μ is small

$$a_{DES} = L + 6(1 + \sin\theta_0)L\mu + O(\mu^2) \tag{60}$$

$$a_{island} = L + 12L\mu + O(\mu^2) \tag{61}$$

$$S_{DES} = S_{QES} \tag{62}$$

- DES proposal and QES proposal gives the same EE as far as for JT gravity.

Conclusion:

- JT gravity can be obtained by dimension reduction in Karch Randall braneworld.
- DES is the holographic counterpart of island formula in JT gravity

Thank you for your attention